1. Fundamental solution

The function $\Phi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2, \\ \frac{1}{(n-2)\omega_n |x|^{n-2}} & n \geqslant 3, \end{cases}$$

is called the fundamental solution for the Laplace operator, where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$. For example, $\omega_3 = 4\pi$.

We can easily check that $\Delta\Phi(x) = 0$ holds for $x \neq 0$.

2. Newtonian potential

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we call the following function the Newtonian potential of f.

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy. \tag{1}$$

Theorem 1. *Suppose that a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x) = 0$ if $|x| \geqslant M$ for some constant M. Then, the potential function* (1) *satisfies*

$$\Delta u(x) = -f(x). \tag{2}$$

We only consider the $n = 2$ case for the purpose of simplicity. The higher dimensions can be similarly shown.

Proof for the two dimension. By using the substitution $y - x = z$, we have

$$u(x) = \int_{\mathbb{R}^2} \Phi(x-y) f(y) dy = \int_{\mathbb{R}^2} \Phi(z) f(x+z) dz.$$

Hence,

$$\Delta u(x) = \int_{\mathbb{R}^2} \Phi(z) \Delta_x f(x+z) dz = \int_{\mathbb{R}^2} \Phi(z) \Delta_z f(x+z) dz = I_\epsilon + J_\epsilon,$$

where $\epsilon < M$ is a small constant and

$$I_\epsilon = \int_{B_M(0) \backslash B_\epsilon(0)} \Phi(z) \Delta_z f(x+z) dz,$$

$$J_\epsilon = \int_{B_\epsilon(0)} \Phi(z) \Delta_z f(x+z) dz.$$

The integration by parts implies

$$I_\epsilon = K_\epsilon - \int_{B_M(0)\setminus B_\epsilon(0)} \nabla_z\Phi(z)\nabla_z f(x+z)dz.$$

where

$$K_\epsilon = \int_{\partial B_M(0)\cup\partial B_\epsilon(0)} \Phi(z)\partial_\nu f(x+z)dz = \int_{\partial B_\epsilon(0)} \Phi(z)\partial_\nu f(x+z)dz.$$

We again apply the integration by parts.

$$I_\epsilon - K_\epsilon = H_\epsilon + \int_{B_M(0)\setminus B_\epsilon(0)} \Delta_z\Phi(z)f(x+z)dz.$$

where

$$H_\epsilon = -\int_{\partial B_M(0)\cup\partial B_\epsilon(0)} \partial_\nu\Phi(z)f(x+z)dz = -\int_{\partial B_\epsilon(0)} \partial_\nu\Phi(z)f(x+z)dz.$$

Since $\Delta_z\Phi(z) = 0$ for $z \neq 0$, we have

$$\Delta u(x) = J_\epsilon + K_\epsilon + H_\epsilon.$$

Next, by using $|\Delta f| \leqslant C$ we have

$$\lim_{\epsilon\to 0} |J_\epsilon| \leqslant C \lim_{\epsilon\to 0} \int_{B_\epsilon(0)} \Phi(z)dz \leqslant C \lim_{\epsilon\to 0} \int_0^{2\pi}\int_0^{\epsilon} \frac{-\log r}{2\pi} r dr d\theta = 0.$$

Also, $|\nabla f| \leqslant C$ implies

$$\lim_{\epsilon\to 0} |K_\epsilon| \leqslant C \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(0)} \Phi(z)dz = -C \lim_{\epsilon\to 0} \frac{2\pi\epsilon\log\epsilon}{2\pi} = 0.$$

On the other hand, on $\partial B_\epsilon(0)$ we have

$$\partial_\nu\Phi(z) = \langle -\frac{z}{2\pi|z|^2}, \nu\rangle = \frac{|z|^2}{2\pi|z|^3} = \frac{1}{2\pi\epsilon}.$$

Therefore,

$$f(x) + \lim_{\epsilon\to 0} H_\epsilon = \int_{\partial B_\epsilon(0)} \frac{f(x)}{2\pi\epsilon}dz - \lim_{\epsilon\to 0}\int_{\partial B_\epsilon(0)} \frac{f(x+z)}{2\pi\epsilon}dz = \lim_{\epsilon\to 0}\int_{\partial B_\epsilon(0)} \frac{f(x)-f(x+z)}{2\pi\epsilon}dz = 0.$$

□

3. Green function

We assume that there exists a smooth function $\varphi : \Omega \times \Omega \to \mathbb{R}$ such that for each fixed $x \in \Omega$ the following hold

$$\Delta_y \varphi(x,y) = 0 \qquad y \in \Omega,$$
$$\varphi(x,y) = \Phi(x-y) \qquad y \in \partial\Omega.$$

Then, the function

$$G(x,y) = \Phi(x,y) - \varphi(x,y)$$

is called the Green function. One can easily check $G(x,y) > 0$ for $x, y \in \Omega$ by the maximum principle. Also, we will show $G(x,y) = G(y,x)$ as an assignment in the problem set 3.

In particular, if $\Omega = B_R(0)$ then

$$G(x,y) = \begin{cases} -\frac{1}{2\pi}\left(\log|x-y| - \log\left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|\right) & n = 2, \\ \frac{1}{(n-2)\omega_n}\left(|x-y|^{2-n} - \left|\frac{R}{|x|}x - \frac{|x|}{R}y\right|^{2-n}\right) & n \geqslant 3. \end{cases}$$

Therefore, the Green's representation formula (6) yields the Poisson's formula

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R} \int_{\partial B_R(0)} \frac{u(y)}{|x-y|^n} dy. \tag{3}$$

In order to show the Green's representation formula, we need the following lemma.

Lemma 2. *Given a smooth bounded domain Ω and a smooth function u, the following holds*

$$u(x) = -\int_\Omega \Phi(x-y)\Delta u(y)dy + \int_{\partial\Omega} \Phi(x-y)u_\nu(y) - u(y)\Phi_\nu(x-y)dy. \tag{4}$$

Proof. The integration by parts provides the Green's identity for smooth functions u, v;

$$\int_\Omega (v\Delta u - u\Delta v)dy = \int_{\partial\Omega} v\partial_\nu u - u\partial_\nu v dy. \tag{5}$$

Given a fixed $x \in \Omega$ we define $\Omega_\epsilon = \Omega \backslash B_\epsilon(x)$ for small ϵ. In addition, we define

$$J_\epsilon = \int_{B_\epsilon(x)} \Phi(x-y)\Delta u(y)dy,$$
$$K_\epsilon = \int_{\partial B_\epsilon(x)} \Phi(x-y)\partial_\nu u(y)dy,$$
$$H_\epsilon = -\int_{\partial B_\epsilon(x)} u(y)\partial_\nu \Phi(x-y)dy.$$

Then, (5) and $\Delta_y \Phi(x-y) = 0$ in Ω_ϵ yield

$$\int_{\Omega_\epsilon} \Phi(x-y)\Delta u(y)dy = \int_{\partial\Omega_\epsilon} \Phi(x-y)\partial_\nu u(y) - u(y)\partial_\nu \Phi(x-y)dy.$$

Therefore, it is enough to show $\lim_{\epsilon\to 0} J_\epsilon + K_\epsilon + H_\epsilon = -u(x)$. By using $|\Delta u| \leqslant C$

$$\lim_{\epsilon\to 0} |J_\epsilon| \leqslant C \lim_{\epsilon\to 0} \int_{B_\epsilon(x)} \Phi(x-y)dy = C \lim_{\epsilon\to 0} \int_{B_\epsilon(0)} \Phi(z)dz = 0.$$

Also, $|\nabla u| \leqslant C$ implies

$$\lim_{\epsilon\to 0} |K_\epsilon| \leqslant C \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} \Phi(x-y)dy = C \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(0)} \Phi(z)dz = 0.$$

On the other hand, on $\partial B_\epsilon(x)$ we have

$$\partial_\nu \Phi(x-y) = \frac{1}{\omega_n |x-y|^{n-1}} = \frac{1}{\omega_n \epsilon^{n-1}} = \frac{1}{|B_\epsilon|}.$$

where $|B_\epsilon|$ is the surface area of a ball of radius ϵ in $\mathbb{R}^n$. Therefore,

$$u(x) + \lim_{\epsilon\to 0} H_\epsilon = \int_{\partial B_\epsilon(x)} \frac{u(x)}{|B_\epsilon|}dy - \lim_{\epsilon\to 0} \int_{\partial B_\epsilon(x)} \frac{u(y)}{|B_\epsilon|}dy = 0.$$

□

Theorem 3 (Green's representation formula)**.** *Suppose that Ω is a smooth bounded domain and u is a smooth function satisfying*

$$\Delta u(x) = f(x) \qquad x \in \Omega,$$
$$u(x) = g(x) \qquad x \in \partial\Omega.$$

Then, the function $u(x)$ satisfies

$$u(x) = -\int_{\Omega} f(y)G(x,y)dy - \int_{\partial\Omega} g(y)G_{\nu}(x,y)dy. \tag{6}$$

Proof. We apply the Green's identity (5) for $\varphi(x,y)$. Then, $\Delta_y\varphi(x,y) = 0$ implies

$$\int_{\Omega} \varphi(x,y)\Delta u(u)dy = \int_{\partial\Omega} \varphi(x,y)\partial_{\nu}u(y) - u\partial_{\nu}\varphi(x,y)dy.$$

Combining with Lemma 2 yields the Green's representation formula. □

4. Harnack's inequality

Theorem 4. *Suppose that $\Delta u = 0$ and $u \geqslant 0$ hold in Ω. Then, for $B_R(z) \subset \Omega$ and $x \in \overline{B}_R(z)$ the following holds*

$$\frac{R^{n-2}(R-r)}{(R+r)^{n-1}}u(z) \leqslant u(x) \leqslant \frac{R^{n-2}(R+r)}{(R-r)^{n-1}}u(z). \tag{7}$$

where $r = |x - z|$.

Proof. Without loss of generality, we assume $z = 0$ and thus $r = |x|$. Then, we recall the Poisson's formula

$$u(x) = \frac{R^2 - |x|^2}{\omega_n R}\int_{\partial B_R(0)} \frac{u(y)}{|x-y|^n}dy.$$

Since $R - |x| \leqslant |x - y| \leqslant R + |x|$ and $|x| = r$, we have

$$\frac{(R-r)}{\omega_n R(R+r)^{n-1}}\int_{\partial B_R(0)} u(y)dy \leqslant u(x) \leqslant \frac{(R+r)}{\omega_n R(R-r)^{n-1}}\int_{\partial B_R(0)} u(y)dy. \tag{8}$$

Hence, the mean value property yields the desired result. □